Limitations on a Generalized Velocity Distribution

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The usefulness of the Pai power series representation for the velocity distribution has been limited because of the lack of knowledge about the functional dependence of the empirical integer constant on the Reynolds number. Correlations are given for this constant for both Newtonian and non-Newtonian materials; however, certain limitations are found as a result of actual velocity distribution calculations.

The equation represents the distribution over most of the turbulent core at all Reynolds numbers, but deviates widely at about a y^+ of 75 for Reynolds numbers greater than 100,000. Below this, the equation provides a satisfactory representation over the entire flow field.

The desirability of a single equation for the description of the turbulent flow velocity profile has long been recognized. The closest approach to this goal has been the semi-empirical method of Pai (*9-11*). An extension of this approach was made by Brodkey, Lee, and Chase (*2*) in which a single equation valid from the wall to the center line, from laminar through turbulent flow, and for Newtonian and non-Newtonian power law materials was

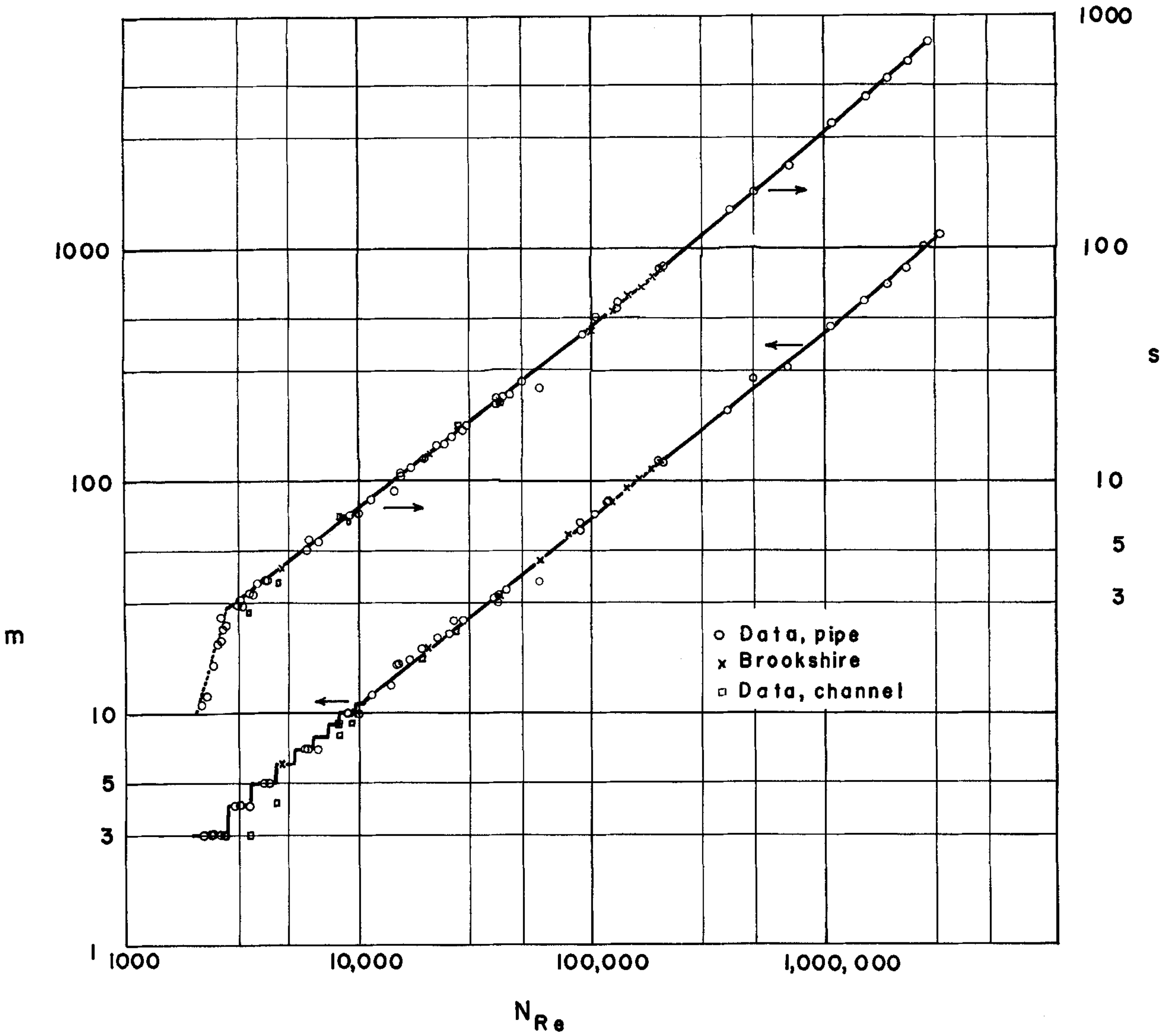

Fig. 1. Correlation of the system characteristic parameter *s* and the integer constant *m* for newtonian fluids.

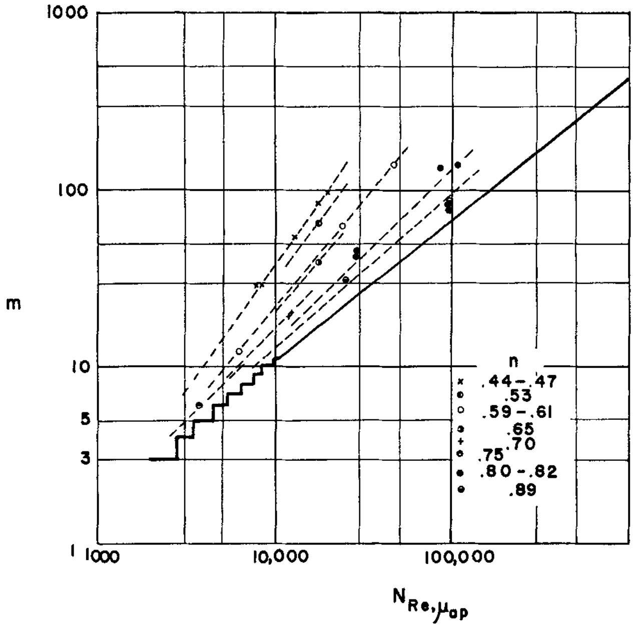

Fig. 2. Correlation of m for non-Newtonian fluids.

derived and tested. The main disadvantage of this and Pai's approach is the necessity of the evaluation of an integer parameter m for each Reynolds number considered. The present paper provides a correlation of this parameter for Newtonian materials and presents certain values for non-Newtonians. In addition, a simple method of evaluating the system characteristic parameter s is given, and comparisons are made with velocity profiles available in the literature.

The derivation of the pertinent equations can be found in reference 2. The final equations of interest in this work are, for pipe flow

$$v/v_{\max} = 1 + a_1\,(r/r_o)^{(n+1)/n} + a_2\,(r/r_o)^{2m} \quad (1)$$

where

$$a_1 = \frac{s-m}{m-(n+1)/2n},\quad a_2 = \frac{(n+1)/2n - s}{m-(n+1)/2n} \quad (2)$$

and

$$\tau_{\text{lam}} = K(-\,dv/dr)^n$$

defines n and K

$$s = (\rho U^{*2}/K)^{1/n}\,(r_o/2v_{\max}) = (y_o^+)^{1/n}/2u_o^+ \quad (3)$$

in which

$$y_o^+ = r_o^n\,U^{*2-n}\,\rho/K,\ u_o^+ = v_{\max}/U^*,\ U^* = \sqrt{\tau_w/\rho},$$

$$\text{and } \tau_w = -\frac{r_o}{2}\frac{dp}{dL} \quad (4)$$

$$v_{\text{avg}}/v_{\max} = 1 + 2a_1n/(3n+1) + a_2/(m+1) \quad (5)$$

When Newtonian materials are considered, the above equations are valid with $n = 1$.

For the Newtonian system, the eddy diffusivity can be easily determined. From reference 2, an integrated form of the Reynolds equation is

$$U^{*2}\,(r/r_o) = -\,\nu(dv/dr) + \overline{v'v'_r} \quad (6)$$

This can be combined with the definition of the eddy diffusivity

$$\overline{v'v'_r} = -\,\epsilon(dv/dr) \quad (7)$$

to give

$$\epsilon/\nu = \frac{-\,U^{*2}\,(r/r_o)}{\nu(dv/dr)} - 1 \quad (8)$$

With n = 1, differentiating Equation (1), using Equation (3) and the above expression (8), one can obtain

$$\epsilon/\nu = \frac{-\,s}{a_1 + ma_2(r/r_o)^{2m-2}} - 1 \quad (9)$$

for the eddy diffusivity, where a_1 and a_2 are given by Equation (2), with $n = 1$. It is interesting to note the limit of this equation at the wall and the center line. At the wall, $r = r_o$, and the limit is zero as expected. At the center line, the limit is
$\epsilon/\nu = -\,(s/a_1) - 1 = (m - sm)/(s - m) = ma_2/a_1 \neq 0$
which is in agreement with Lynn (*17*) but not with Gill and Scher (*4*). An expression for the eddy diffusivity can be developed for the non-Newtonian fluid case; however, it is more complex and does not reduce to a simple form as in Equation (9).

The preceding equations, except for Equation (5), are valid for parallel plates as well as for tubes with the minor change that r and r_o are replaced with the distance from the center line, that is, y and y_o. Equation (5), for parallel plates, becomes

$$v_{\text{avg}}/v_{\max} = 1 + a_1n/(2n+1) + a_2/(2m+1) \quad (10)$$

The literature (*1, 3, 6-8, 12, 16, 18-20*) contains many evaluations of $v_{\text{avg}}/v_{\max}$ at various Reynolds numbers. From these, unique values of m and s were determined as a function of the Reynolds number by the following procedure:

1. s was calculated from Equations (3) and (4) with the aid of the relation

$$y_o^+ = \sqrt{f/8}\,N_{Re} \quad (11)$$

which was used if needed.

2. Equation (5) [or (10)] contains the variables $v_{\text{avg}}/v_{\max}$, s, and m. Since the first two variables are known, m was obtained by a simple trial-and-error solution.

The results of the computation for s for Newtonian materials are shown in Figure 1 and were expressed as the following correlation equations:

$$s = 0.585 + 3.172 \times 10^{-3}\,(N_{Re})^{0.833}\ N_{Re} > 2{,}800 \quad (12)$$

$$s = 2.417 \times 10^{-12}\,(N_{Re})^{3.51}\ 2{,}040 < N_{Re} < 2{,}800 \quad (13)$$

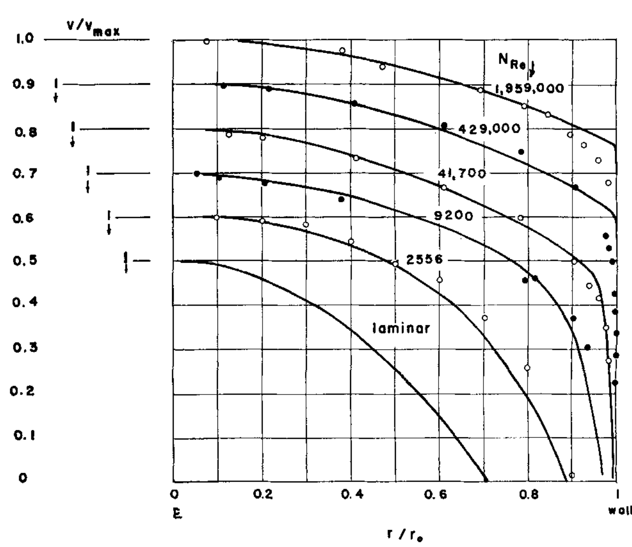

Fig. 3. Velocity distributions.

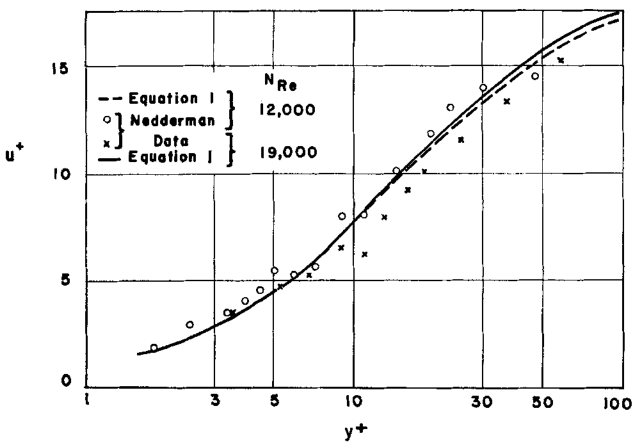

Fig. 4. Near the wall test with Nedderman's data.

$$s = 1 \qquad N_{Re} < 2{,}040 \quad (14)$$

Equation (12) had an average percentage deviation of 2.3% and Equation (13), 5.1%. The results for m are also shown in Figure 1 and can be expressed as the whole integer closest to the value of m as given by

$$m = -0.617 + 8.211 \times 10^{-3} (N_{Re})^{0.786} \quad (15)$$

This equation had an average percentage deviation of 2.9%. The equations for s and m were obtained by using the extensive pipe-flow data available in the literature. Several parallel plate runs from reference *18* to *20* were also calculated; these are shown in Figure 1. At higher Reynolds numbers (based on the average velocity and the total width of the channel) the points agree with the equations. At very low Reynolds numbers, the value of m from Equation (15) would be one unit too high.

Equation (15) is the more important result of this analysis, since it provides a means of estimating the parameter m, heretofore estimable only from velocity-distribution data. s can be estimated from Equation (3); however, if only the Reynolds number is known, then Equations (12) to (14) provide the only means for obtaining s, since v_{avg} is unknown. The unique correlation of s with the Reynolds number results from all terms in Equation (3) also being unique functions of the Reynolds number or constants of the system. The same applies to m's evaluation from Equation (5) [or (10)], since s is now known to be a unique function of the Reynolds number.

Brookshire (3) has independently arrived at similar results; that is, he showed that for Newtonian materials m could be determined from average data from

$$m = \frac{(f/16)\, N_{Re} - 1}{1 - (v_{max}/2v_{avg})} \quad (16)$$

His calculated values of m and s have been added to Figure 1. They were not included in the determination of the correlation equation; however, the excellent agreement is apparent. In effect, the two methods are identical, and for Newtonian materials Brookshire's method of evaluation is more convenient than the trial-and-error solution used here.

A number of values of m were calculated by the trial-and-error procedure for non-Newtonian flows from the literature data (*1, 14*). These are shown in Figure 2 with the Newtonian correlation curve. The possible correlation is shown; however, more data is needed before a satisfactory correlation is possible; thus, no further comparisons beyond that reported in reference 2 are reported here for the non-Newtonian materials.

In the above analysis, the values of m are not based on a fit to a velocity profile, but rather are estimated from average data over the cross section. Thus, to establish their true worth one must compare the equations with actual velocity profile data. To be valuable, the equations, with the now known value of m, must be able to predict satisfactorily the entire velocity profile. Such a comparison was made and is presented below; Brookshire independently made a similar comparison and obtained essentially the same results. Figure 3 is a plot of v/v_{max} vs. r/r_o. The equation predicts accurately the major part of the distribution over the central core for all Reynolds numbers; however, deviation near the wall is apparent.

More information can be gained from an analysis of the u^+ vs. y^+ plots, which emphasize the wall area. At low Reynolds numbers and near the wall, the comparison is quite good with the recent data of Nedderman (7) (see Figure 4). This is an excellent confirmation of the equation in this range of Reynolds numbers.

Since these results and Pai's (*11*) comparison with Laufer's data (*6*) were quite good, an extensive calculation of distributions was made for the range of Reynolds numbers from laminar flow to 6×10^6. The major deviation from the log profile equation occurs around a y^+ of about 75; it is about 9% high for a Reynolds number of 100,000 and increases rapidly as the Reynolds number increases. Figures 5 and 6 show some actual comparisons with pipe-flow experimental data. Above the 100,000 Reynolds number value, the inadequacy of the representation at the y^+ of 75 area is apparent. This is the location of the shoulder of the profiles shown in Figure 3. The Reynolds number value of 100,000 is the upper usable limit of Pai's equation over the entire pipe. Similar results were obtained with channel flow.

With the addition of Equation (15), the velocity-distribution equation developed by Pai (Equations (1), (2),

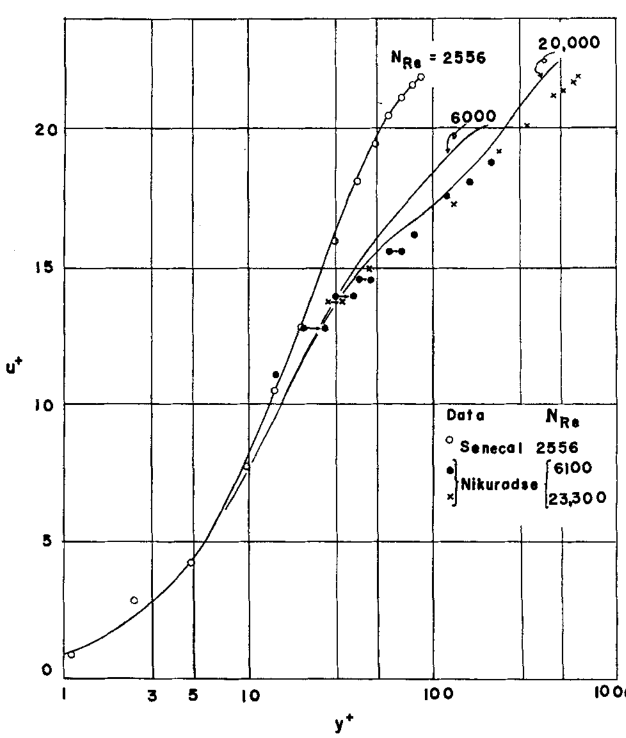

Fig. 5. Comparison with data for low Reynolds numbers.

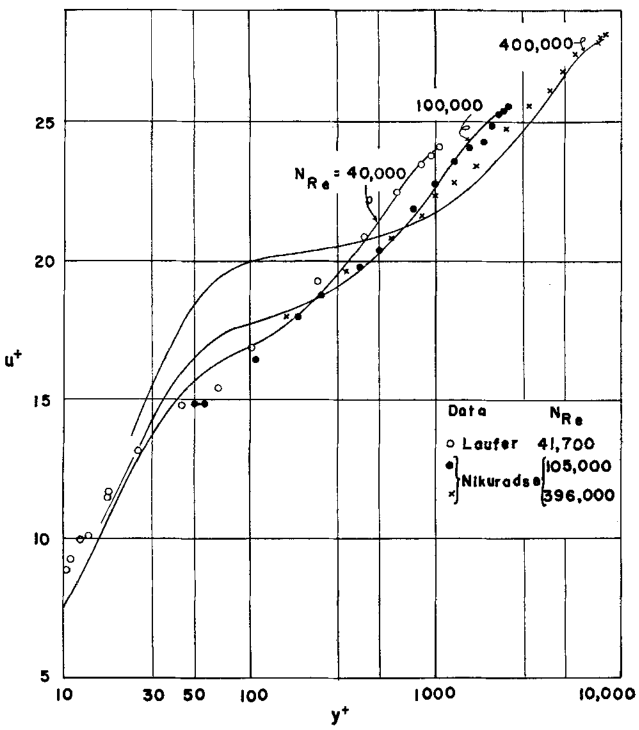

Fig. 6. Comparison with data for high Reynolds numbers.

and (3) with $n = 1$) is completely determined for any Reynolds number. Below $N_{Re} = 100{,}000$ the representation is quite good and gives a very convenient equation which is valid from the wall to the center line and from laminar through turbulent flow.

For non-Newtonian flow, the constant corresponding to that obtained from Equation (15) has been evaluated (see Figure 2). Preliminary calculations indicate that the equation does represent the velocity distribution when the estimated value of m and the proper value of s are used.

Recently two additional attempts have been reported for a possible universal velocity distribution for Newtonian materials. Spalding (*15*) suggested a totally empirical formula which is valid for turbulent flow only, and cannot be used in the transition region. Furthermore, the equation approaches the standard logarithmic form at high value of y^+ and thus cannot satisfy the boundary condition of symmetry at the center line. Another semi-empirical formula was suggested by Gill and Scher (*4*) and applied to a heat transfer problem by Gill and Lee (*5*). The use of this equation requires a numerical solution and a graphical representation. Thus, it cannot be used as a simple analytical representation for the velocity in closed form; however, if a computor solution is to be used, it could be programmed into the problem.

ACKNOWLEDGMENT

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NOTATION

a_1, a_2 = constants
f = Fanning friction factor
K = defined by Equation (2)
m = empirical constant, an integer
n = defined by Equation (2)
$N_{Re,\mu ap}$ = $Dv_{avg}\,\rho/\mu_{ap}$ based on the apparent viscosity obtained from capillary data, μ_{ap}
Q = volumetric flow rate
r = radius
r_o = pipe radius
s = defined by Equation (3)
U^* = $\sqrt{\tau_w/\rho}$, friction velocity
v = velocity, average at a point
v_{avg} = velocity, average over pipe
v_{max} = velocity, maximum
y_o^+ = defined by Equation (4)
u_o^+ = defined by Equation (4)

Greek Letters

τ_w = wall shear stress
τ_{lam} = laminar shear stress
π = 3.1416
μ_{ap} = apparent viscosity as measured on a capillary shear diagram ($4Q/\pi r_o^3$ vs. $r_o \Delta p/2L$)
ρ = density
ν = μ_{2p}/ρ
ϵ = eddy diffusivity

Subscripts

w = wall

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